Legendre polynomials used for zonal and tesseral geoids

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1 Purpose of the analysis

This paper analyses the Legendre decomposition [R 1] and its use in the earth gravitational potential.

2 Formulation

For a given x, the formulation of the Legendre polynomials used in the decomposition is given by the nth derivative of the polynomials $(x^2 - 1)^n$ with some scaling factor.

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n (x^2-1)^n}{dx^n} \qquad [1]$$

Using this polynomial, an additional mth derivative allows to define the further Legendre polynomials.

$$P_{n,m}(x) = (1 - x^2)^{\frac{m}{2}} . \frac{d^m P_n(x)}{dx^m} \qquad [2]$$

By stating $\frac{d^0 P(x)}{dx^0} = P(x)$ i.e. there are no zero-derivation, one can see that $P_n(x) = P_{n,0}(x)$ which can make the second equation *[2]* the unique definition.

$$P_{n,m}(x) = \frac{(1-x^2)^{\frac{m}{2}}}{2^n\, n!} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} \qquad n, m = 0, \infty$$

3 Explicit Legendre polynomials

Using Mathcad, it is rather simple to differentiate any equation, and factorize the result to make it clear. This has been done in this chapter.

Note: For the gravitational potential one use $x = sin(\varphi)$ with φ the latitude, and because $\varphi \in [-90°, °90°]$ then $cos(\varphi)$ is always >=0, so it is correct to write $\sqrt{1 - sin^2(\varphi)} = cos(\varphi)$.

Sometimes, people use the co-latitude $x = cos(\theta)$. Of course, it is correct to write $\sqrt{1 - cos^2(\theta)} = sin(\theta)$ because it is always >=0 for $\theta \in [0°, 180°]$.

Legendre Polynomials, *using Mathcad*

$$f(x) = \left[\frac{1}{2^n \cdot n!}\left(x^2-1\right)^n\right]$$

For n=0, $P_0(x)=1$

n = 1: $f(x) = \frac{1}{2}\cdot x^2 - \frac{1}{2}$; $\frac{d}{dx}f(x) = x$

n = 2: $f(x) = \frac{1}{8}\cdot\left(x^2-1\right)^2$; $\frac{d}{dx}f(x) = \frac{1}{2}\cdot\left(x^2-1\right)\cdot x$; $\frac{d}{dx}\frac{d}{dx}f(x) = \frac{3}{2}\cdot x^2 - \frac{1}{2}$

n = 3: $f(x) = \frac{1}{48}\cdot\left(x^2-1\right)^3$; $\frac{d}{dx}f(x) = \frac{1}{8}\cdot\left(x^2-1\right)^2\cdot x$; $\frac{d}{dx}\frac{d}{dx}f(x) = \frac{1}{2}\cdot\left(x^2-1\right)\cdot x^2 + \frac{1}{8}\cdot\left(x^2-1\right)^2$; $\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}f(x) = x^3 + 2\cdot\left(\frac{1}{2}\cdot x^2 - \frac{1}{2}\right)\cdot x + \frac{1}{2}\cdot\left(x^2-1\right)\cdot x$

n = 4: $f(x) = \frac{1}{384}\cdot\left(x^2-1\right)^4$; $\frac{d}{dx}f(x) = \frac{1}{48}\cdot\left(x^2-1\right)^3\cdot x$; $\frac{d}{dx}\frac{d}{dx}f(x) = \frac{1}{8}\cdot\left(x^2-1\right)^2\cdot x^2 + \frac{1}{48}\cdot\left(x^2-1\right)^3$; $\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}f(x) = \frac{1}{2}\cdot\left(x^2-1\right)\cdot x^3 + \frac{3}{8}\cdot\left(x^2-1\right)^2\cdot x$

$$\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}f(x) = x^4 + 3\cdot\left(\frac{1}{2}\cdot x^2 - \frac{1}{2}\right)\cdot x^2 + \frac{3}{2}\cdot\left(x^2-1\right)\cdot x^2 + \frac{3}{8}\cdot\left(x^2-1\right)^2$$

$P_1(x) = x$ $\quad P_2(x) = \frac{3}{2}\cdot x^2 - \frac{1}{2}$ $\quad P_3(x) = \frac{1}{2}\cdot x\cdot\left(5\cdot x^2 - 3\right)$ $\quad P_4(x) = \frac{35}{8}\cdot x^4 - \frac{15}{4}\cdot x^2 + \frac{3}{8}$ $\quad P_0(x)=1$

For the gravitational potential, φ being the latitude $x = \sin(\varphi)$

$P_1(x) = \sin(\varphi)$ $\quad P_2(x) = \frac{3}{2}\cdot\sin(\varphi)^2 - \frac{1}{2}$ $\quad P_3(x) = \frac{1}{2}\cdot\sin(\varphi)\cdot\left(5\cdot\sin(\varphi)^2 - 3\right)$ $\quad P_4(x) = \frac{35}{8}\cdot\sin(\varphi)^4 - \frac{15}{4}\cdot\sin(\varphi)^2 + \frac{3}{8}$ $\quad P_0(x)=1$

	$\left(1-x^2\right)^{\frac{m}{2}}$	derivative ***m-1*** of $P_n(x)$	derivative ***m*** of $P_n(x)$	**Polynomial** $P_{n,m}(x)$	
m = 1	$\sqrt{1-x^2}$	$P1(x) = x$	$P1(x) = 1$	$P_{1,1}(x) = \sqrt{1-x^2}$	$P_{1,1}(x) = \cos(\varphi)$
		$P2(x) = \frac{3}{2}\cdot x^2 - \frac{1}{2}$	$P2(x) = 3\cdot x$	$P_{2,1}(x) = 3\cdot x\cdot\sqrt{1-x^2}$	$P_2,1(x) = 3\cdot\sin(\varphi)\cos(\varphi)$
		$P3(x) = \frac{1}{2}\cdot x\cdot\left(5\cdot x^2-3\right)$	$P3(x) = \frac{15}{2}\cdot x^2 - \frac{3}{2}$	$P_{3,1}(x) = \left(\frac{15}{2}\cdot x^2 - \frac{3}{2}\right)\cdot\sqrt{1-x^2}$	$P_{3,1}(x) = \left(\frac{15}{2}\cdot\sin(\varphi)^2 - \frac{3}{2}\right)\cdot\cos(\varphi)$
		$P4(x) = \frac{35}{8}\cdot x^4 - \frac{15}{4}\cdot x^2 + \frac{3}{8}$	$P4(x) = \frac{35}{2}\cdot x^3 - \frac{15}{2}\cdot x$	$P_{4,1}(x) = \left(\frac{35}{2}\cdot x^3 - \frac{15}{2}\cdot x\right)\cdot\sqrt{1-x^2}$	$P_{4,1}(x) = \left(\frac{35}{2}\cdot\sin(\varphi)^3 - \frac{15}{2}\cdot\sin(\varphi)\right)\cdot\cos(\varphi)$
m = 2	$1-x^2$	$P2(x) = 3\cdot x$	$P2(x) = 3$	$P_{2,2} = 3\left(1-x^2\right)$	$P_{2,2} = 3\left(\cos(\varphi)\right)^2$
		$P3(x) = \frac{15}{2}\cdot x^2 - \frac{3}{2}$	$P3(x) = 15x$	$P_{3,2} = 15x\left(1-x^2\right)$	$P_{3,2} = 15\sin(\varphi)\cdot\left(\cos(\varphi)\right)^2$
		$P4(x) = \frac{35}{2}\cdot x^3 - \frac{15}{2}\cdot x$	$P4(x) = \frac{105}{2}\cdot x^2 - \frac{15}{2}$	$P_{4,2} = \left(\frac{105}{2}\cdot x^2 - \frac{15}{2}\right)\cdot\left(1-x^2\right)$	$P_{4,2} = \left(\frac{105}{2}\cdot\sin(\varphi)^2 - \frac{15}{2}\right)\cdot\left(\cos(\varphi)\right)^2$
m = 3	$\left(\sqrt{1-x^2}\right)^3$	$P3(x) = 15\cdot x$	$P3(x) = 15$	$P_{3,3} = 15\left(\sqrt{1-x^2}\right)^3$	$P_{3,3} = 15\cdot\left(\cos(\varphi)\right)^3$
		$P4(x) = \frac{105}{2}\cdot x^2 - \frac{15}{2}$	$P4(x) = 105x$	$P_{4,3} = 105\cdot x\left(\sqrt{1-x^2}\right)^3$	$P_{4,3} = 105\cdot\sin(\varphi)\cdot\left(\cos(\varphi)\right)^3$
m = 4	$\left(1-x^2\right)^2$	$P4(x) = 105\cdot x$	$P4(x) = 105$	$P_{4,4} = 105\left(1-x^2\right)^2$	$P_{4,4} = 105\cdot\left(\cos(\varphi)\right)^4$

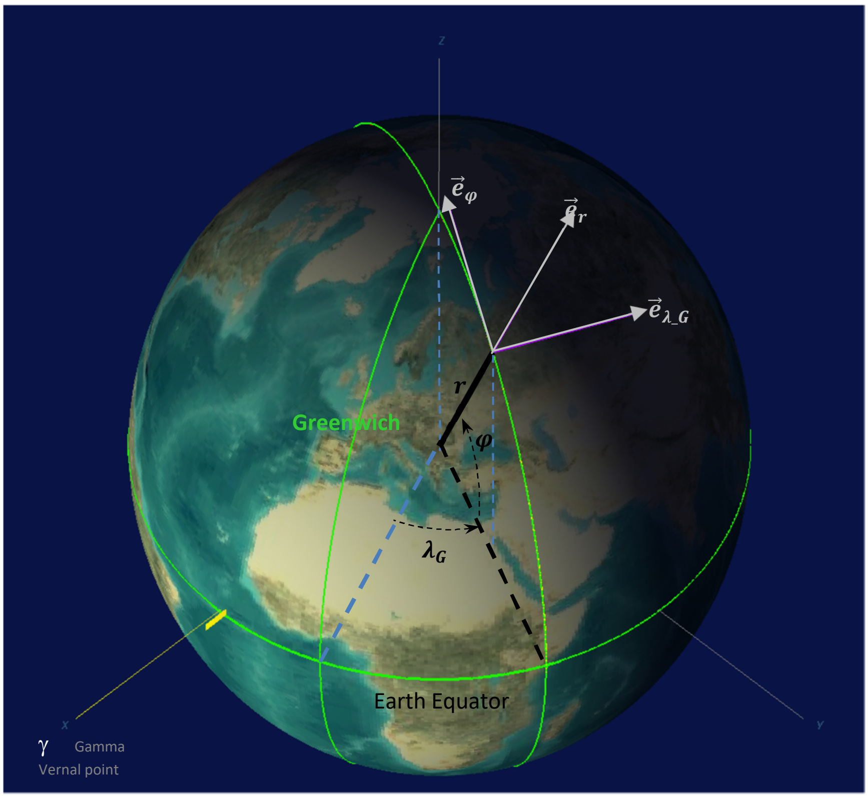
There exists a recurrence with $P_{0,0}(x) = 1$; $P_{0,1}(x) = 0$; $P_{1,0}(x) = x$; $P_{1,1}(x) = \sqrt{1-x^2}$

for $m > n$ $\quad P_{n,m}(x) = 0;$ $\qquad$ for $m \le n$, $\; P_{n+1,m}(x) = \frac{x(2n+1)P_{n,m}(x) - (n+m)P_{n-1,m}(x)}{n+1-m}$

$$P_{n,m+2}(x) = \frac{x}{\sqrt{1-x^2}}(2m+2)P_{n,m+1}(x) - \left(n^2 - m^2 + m - n\right)P_{n,m}(x)$$

4 Application to the Earth potential

The Earth's external gravitational potential $V(r,\varphi,\lambda)$ for $r > R_e$ can be developed into a series of zonal spherical harmonics and sectorial & tesseral spherical harmonics.



Forces are given by $\frac{\vec{F}}{M} = -\overrightarrow{\nabla V}$.

Note : in geodesy the "potential" $U = -V$ is defined with positive sign which is opposite to the definition in physics as shown above. So for about 50% of the references the forces are given by $+\overrightarrow{\nabla U}$.

The Earth's external gravitational potential V vary from zero at an infinite location to a negative value at the Earth surface.

$$V(r,\lambda,\varphi) = -\frac{G \cdot m_e}{r}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n P_{n,m}(\sin(\varphi)).\left[C_{n,m}\cos(m\lambda) + S_{n,m}\sin(m\lambda)\right]\right]$$

with
- r : the distance to the Earth centre of mass,
- φ : the latitude from the equator toward the North, $[-90, +90°]$
- λ_G the East longitude with respect to the Greenwich meridian
- R_e the equatorial radius of the Earth
- $G.m_e$ the Earth gravitational constant

The linear combination "$C_{n,m}\cos(m\lambda) + S_{n,m}\sin(m\lambda)$ " can be easily made more meaningful using "$J_{n,m} * \cos(m\lambda - m\Lambda) = J_{n,m} * \cos(m\lambda)\cos(m\Lambda) + J_{n,m} * \sin(m\lambda)\sin(m\Lambda)$".

- For $m = 0$, it is clear that $J_{n,0} = C_{n,0}$.
- For $m > 0$, by identification one gets: $C_{n,m} = J_{n,m} * \cos(m\Lambda_{n,m})$ and $S_{n,m} = J_{n,m} * \sin(m\Lambda_{n,m})$ leading to $J_{n,m} = \pm\sqrt{C_{n,m}^2 + S_{n,m}^2}$ and $\Lambda_{n,m} = \frac{1}{m}\text{atan2}(\frac{S_{n,m}}{J_{n,m}}, \frac{C_{n,m}}{J_{n,m}})$. Hence the value of $J_{n,m}$ depends on the chosen determination, which further determines $\Lambda_{n,m}$.

One defines $\lambda_{n,m} = \frac{1}{m}\text{atan}\left(\frac{S_{n,m}}{C_{n,m}}\right)$ which is of course not always$=\Lambda_{n,m}$.

- For $m = 0$, one defines the zonal term J_n using a negative rule $J_n = -J_{n,0}$ i.e. $J_n = -C_{n,0}$, hence the corresponding terms in the potential shall be corrected by a minus sign. This seems to be adopted by everybody, for example $J_2 = +0.001082$, $J_3 = -2.534E-6$.
- for $m > 0$, some people change the rule and take the positive root $J_{n,m} = +\sqrt{C_{n,m}^2 + S_{n,m}^2}$, the angle is always $\Lambda_{n,m} = \lambda_{n,m} = \frac{1}{m}\text{atan}(\frac{S_{n,m}}{C_{n,m}})$.

  While other still use the same negative root rule, so $J_{n,m} = -\sqrt{C_{n,m}^2 + S_{n,m}^2}$, but in such case, the value for the angle $\Lambda_{n,m}$ is $\pi + \lambda_{n,m}$.

  Because $\cos(m\lambda - m\Lambda_{n,m}) = \cos\left(m\lambda - \pi - \frac{1}{m}\text{atan}(\frac{S_{n,m}}{C_{n,m}})\right) = -\cos\left(m\lambda - \frac{1}{m}\text{atan}(\frac{S_{n,m}}{C_{n,m}})\right)$, the two forms can be made similar except for the sign of the cos() and the sign of $J_{n,m}$.

In this short note, one keeps the negative root rule for all $J_{n,m}$, so with $x = sin(\varphi)$:

$$V(r, \lambda, x) = -\frac{G \cdot m_e}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_n(x) - \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n J_{n,m} P_{n,m}(x) cos(m(\lambda - \lambda_{n,m}))\right]$$

J_n the zonal non dimensional terms = $-C_{n,0}$

$J_{n,m}$ the sectorial & tesseral terms (negative values $J_{n,m} = -\sqrt{C_{n,m}^2 + S_{n,m}^2}$)

$\lambda_{n,m}$ the sect. East long. with respect to Greenwich, $\lambda_{n,m} = \frac{1}{m}\tan^{-1}\left(\frac{S_{n,m}}{C_{n,m}}\right)$

with "n" starting at 2 because the terms for n=0 is +1 (the first term) and for n=1 the terms can be set to zero for a frame centred on the earth centre of gravity and with the Legendre polynomials

$\boldsymbol{P_n}(\boldsymbol{x}) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$ $\quad$ $P_{n,m}(x) = (1-x^2)^{\frac{m}{2}}.\frac{d^m \boldsymbol{P_n}(x)}{dx^m}$ as seen before.

4.1 Conclusion

In order to avoid any confusion with the chosen determination, it however is preferable to use:

$$V(r, \lambda, x) = -\frac{G \cdot m_e}{r}\left[1 - \sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n J_n P_n(x) + \sum_{n=2}^{\infty}\sum_{m=1}^{n}\left(\frac{R_e}{r}\right)^n P_{n,m}(x)[C_{n,m} cos(m\lambda) + S_{n,m} sin(m\lambda)]\right]$$

References:

[R 1] P. Duchon, J.M. Guilbert, L. Marechal, "Stabilisation des satellites," 1983 Supaero.

[R 2] KopooS, TriaXOrbitaL tool 1989-2021

[R 3] Luc Duriez, Cours de Mécanique céleste classique, Laboratoire d'Astronomie de l'Université de Lille 1 et IMCCE de l'Observatoire de Paris, 2007

La mécanique orbitale est une discipline étrange... La première fois que vous la découvrez, vous ne comprenez rien... La deuxième fois, vous pensez que vous comprenez, sauf un ou deux points.. La troisième fois, vous savez que vous ne comprenez plus rien, mais à ce niveau vous êtes tellement habitué que ça ne vous dérange plus. attribué à Arnold Sommerfeld pour la thermodynamique, vers 1940
