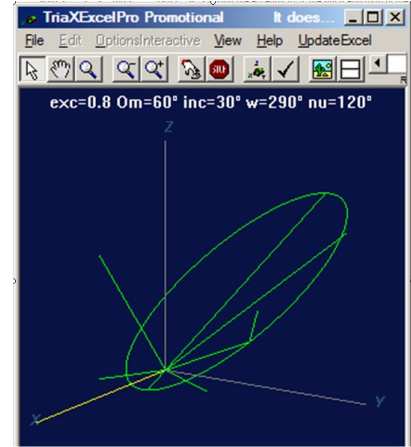
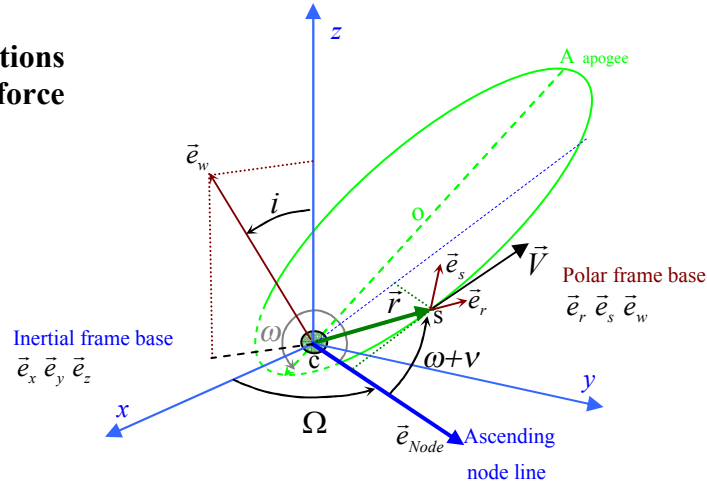


Trajectory equations under a central force


The trajectory under a central force $\ddot{\vec{r}} = \frac{-\mu}{r^3} \vec{r}$ is fixed at each time t by the vectors $\vec{r}(t), \vec{V}(t)$.

With $\vec{r}(t), \vec{V}(t)$ one can derive the 5 fixed elliptical parameters: a, e, i, Ω, ω and one time dependant $v(t)$

It is fully known that the vis viva equation (orbital energy conservation) is $\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$ which give the semi-major axis a .

With $\vec{C} = \vec{r} \wedge \vec{V}$ and $C^2 = \mu a(1 - e^2)$ giving the eccentricity e , the polar equation of the ellipse $r(1 + e \cos v) = a(1 - e^2)$ and its radial derivative $\mu e r \sin v = C \vec{V} \cdot \vec{r}$ give the true anomaly $v(t)$. The angles of inclination $\in [0, \pi[$ and right ascension of the ascending node i, Ω are defined by the vector \vec{C} ($\vec{C} \cdot \vec{e}_z = C \cos i$; $C \sin \Omega = \vec{C} \cdot \vec{e}_x$; $C \cos \Omega = -\vec{C} \cdot \vec{e}_y$). The perigee argument ω is given by the polar equation projected on the node line and then on the inertial frame:

$$\vec{r} \cdot \vec{e}_x = -r \sin(\omega + v) \cos i \sin \Omega + r \cos(\omega + v) \cos \Omega \quad \vec{r} \cdot \vec{e}_y = +r \sin(\omega + v) \cos i \cos \Omega + r \cos(\omega + v) \sin \Omega$$

$$\vec{r} \cdot \vec{e}_z = +r \sin(\omega + v) \sin i \Rightarrow \vec{r} \cdot \vec{e}_x \cos \Omega + \vec{r} \cdot \vec{e}_y \sin \Omega = r \cos(\omega + v) \quad \text{and} \quad r \sin(\omega + v) = \vec{r} \cdot \vec{e}_z / \sin i$$

The mean anomaly is $M(t) = \sqrt{\frac{\mu}{a^3}} (t - t_{per}) = \sqrt{\frac{\mu}{a^3}} (u(t) - e \sin u(t))$ where the time when crossing the perigee is t_{per} and u is the eccentric anomaly (angle from center O wrt perigee) $r = a(1 - e \cos u(t))$.

Trajectory equations with perturbation force \vec{F} force per unit of mass or acceleration $\ddot{\vec{r}} = \frac{d\vec{V}}{dt} = \frac{-\mu}{r^3} \vec{r} + \vec{F}$

$$\vec{C} = \vec{r} \wedge \vec{V} \quad \frac{d\vec{C}}{dt} = \vec{r} \wedge \vec{F} \quad h = \frac{V^2}{2} - \frac{\mu}{r} \quad \frac{dh}{dt} = \vec{V} \cdot \vec{F} \quad \vec{E} = \frac{1}{\mu} \vec{V} \wedge \vec{C} - \frac{\vec{r}}{r} \quad \frac{d\vec{E}}{dt} = \frac{1}{\mu} [\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F})]$$

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$$

$$\frac{da}{dt} = \frac{2a^2}{\mu} \vec{V} \cdot \vec{F}$$

$$\frac{dC}{dt} = \frac{1}{C} \vec{C} \wedge \vec{r} \cdot \vec{F}$$

$$e^2 = 1 - \frac{C^2}{\mu a}$$

$$\frac{de}{dt} = \frac{C^2}{\mu^2 e} \vec{V} \cdot \vec{F} - \frac{\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu e a}$$

$$\frac{d\vec{E}}{dt} = \frac{1}{\mu e} [\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F})] \cdot \vec{E}$$

$$\vec{C} \cdot \vec{e}_z = C \cos i$$

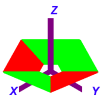
$$\frac{di}{dt} = \frac{r \cos(\omega + v)}{C^2} \vec{C} \cdot \vec{F}$$

$$\text{tg} \Omega = \frac{\vec{C} \cdot \vec{e}_x}{-\vec{C} \cdot \vec{e}_y}$$

$$\frac{d\Omega}{dt} = \frac{r \sin(\omega + v)}{C^2 \sin i} \vec{C} \cdot \vec{F}$$

$$\frac{d\omega}{dt} = \frac{1}{\mu C e^2} [\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F})] \cdot \vec{C} \wedge \vec{E} - \cos i \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} \approx \frac{M(t)}{t - t_{per}} + \frac{-1}{\sqrt{\mu a}} \left[2\vec{r} \cdot \vec{F} + C \frac{d\omega}{dt} + C \cos i \frac{d\Omega}{dt} \right]$$



Application: best command to increase the perigee, [R1]

$$r_{per} = a(1 - e) \quad \frac{dr_{per}}{dt} = (1 - e)\dot{a} - a\dot{e}$$

$$\frac{dr_{per}}{dt} = \frac{\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu e} - \frac{(1 - e)^2 a^2}{\mu e} \vec{V} \cdot \vec{F}$$

To increase the perigee at best, the thrust shall be in the local horizontal (i.e. along $\vec{C} \wedge \vec{r}$ in order to get $\vec{C} \cdot (\vec{r} \wedge \vec{F})$ maximum) and around apogee (i.e. \vec{r} maximum and $\vec{V} \cdot \vec{F}$ minimum).

Application: best command to increase the apogee, [R1]

$$r_{apo} = a(1 + e) \quad \frac{dr_{apo}}{dt} = (1 + e)\dot{a} + a\dot{e}$$

$$\frac{dr_{apo}}{dt} = \frac{(1 + e)^2 a^2}{\mu e} \vec{V} \cdot \vec{F} - \frac{\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu e}$$

To increase the apogee at best, the thrust shall be along velocity (i.e. $\vec{V} \cdot \vec{F}$ maximum) and around perigee (i.e. \vec{V} maximum and $\vec{C} \cdot (\vec{r} \wedge \vec{F})$ minimum).

Development of Gauss equations [R2], [R3], [R4]

Note: all derivative wrt an inertial frame, thus $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are fixed vector $\Rightarrow \dot{\vec{e}}_x, \dot{\vec{e}}_y, \dot{\vec{e}}_z = 0$

$$\ddot{\vec{r}} = \frac{d\vec{V}}{dt} = \frac{-\mu}{r^3} \vec{r} + \vec{F} \quad \text{nota: } \vec{V} = \frac{d\vec{cs}}{dt} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \frac{d(r\vec{e}_r)}{dt} = \frac{dr}{dt} \vec{e}_r + r \frac{d\vec{e}_r}{dt} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\dot{\nu} \vec{e}_s$$

$$\vec{C} = \vec{r} \wedge \vec{V} \quad \frac{d\vec{C}}{dt} = \frac{d\vec{r}}{dt} \wedge \vec{V} + \vec{r} \wedge \frac{d\vec{V}}{dt} = \vec{V} \wedge \vec{V} + \vec{r} \wedge \frac{-\mu}{r^3} \vec{r} + \vec{r} \wedge \vec{F} = 0 + 0 + \vec{r} \wedge \vec{F} \quad \frac{d\vec{C}}{dt} = \vec{r} \wedge \vec{F}$$

$$h = \frac{V^2}{2} - \frac{\mu}{r} \quad \text{for elliptical orbits } h = \frac{-\mu}{2a}$$

$$\text{with } \frac{1}{2} \frac{d\dot{r}^2}{dt} = \ddot{r} \cdot \dot{r} \quad \text{and} \quad \frac{d\frac{\mu}{r}}{dt} = \frac{d\mu(\vec{r} \cdot \vec{r})^{-1/2}}{dt} = \mu \frac{-1}{2} (\vec{r} \cdot \vec{r})^{-3/2} \cdot 2(\vec{r} \cdot \dot{r}) = \frac{-\mu}{r^3} \vec{r} \cdot \dot{r}$$

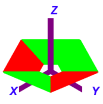
$$\ddot{r} \cdot \dot{r} = \left(\frac{-\mu}{r^3} \vec{r} + \vec{F} \right) \cdot \dot{r} = \frac{-\mu}{r^3} \vec{r} \cdot \dot{r} + \vec{F} \cdot \dot{r} = \frac{d\frac{\mu}{r}}{dt} + \vec{F} \cdot \dot{r} = \frac{d\frac{\mu}{r}}{dt} + \vec{V} \cdot \vec{F}$$

$$\frac{dh}{dt} = \frac{d\left(\frac{\dot{r}^2}{2} - \frac{\mu}{r}\right)}{dt} = \frac{1}{2} \frac{d\dot{r}^2}{dt} - \frac{d\frac{\mu}{r}}{dt} = \vec{V} \cdot \vec{F}$$

$$\frac{dh}{dt} = \vec{V} \cdot \vec{F}$$

$$a = \frac{-\mu}{2h} \quad \frac{da}{dt} = \frac{2a^2}{\mu} \vec{V} \cdot \vec{F}$$

$$\frac{da}{dt} = \frac{\mu}{2h^2} \cdot \frac{dh}{dt} = \frac{\mu}{2h^2} \vec{V} \cdot \vec{F} = \frac{2a^2}{\mu} \vec{V} \cdot \vec{F}$$



$$\vec{E} = \frac{1}{\mu} \vec{V} \wedge \vec{C} - \frac{\vec{r}}{r} \quad \frac{d\vec{E}}{dt} = \frac{1}{\mu} \frac{d\vec{V}}{dt} \wedge \vec{C} + \frac{1}{\mu} \vec{V} \wedge \frac{d\vec{C}}{dt} - \dot{\vec{e}}_r = \frac{1}{\mu} \left[\frac{-\mu}{r^3} \vec{r} + \vec{F} \right] \wedge \vec{C} + \frac{1}{\mu} \vec{V} \wedge (\vec{r} \wedge \vec{F}) - \dot{\vec{e}}_r$$

$$\frac{d\vec{E}}{dt} = \frac{1}{\mu} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] - \left(\frac{\vec{r}}{r^3} \wedge (\vec{r} \wedge \vec{V}) + \dot{\vec{e}}_r \right) = \frac{1}{\mu} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] + 0 \quad \text{because}$$

$$\frac{\vec{r}}{r^3} \wedge (\vec{r} \wedge \vec{V}) + \dot{\vec{e}}_r = \frac{\vec{r} \cdot \vec{V}}{r^3} \vec{r} - \frac{\vec{r} \cdot \vec{r}}{r^3} \vec{V} + \dot{\vec{e}}_r = \frac{\vec{e}_r \cdot \vec{V}}{r} \vec{e}_r - \frac{\mu}{r} \vec{e}_r + \dot{\vec{e}}_r = \frac{\dot{r}}{r} \vec{e}_r + (\vec{e}_r \cdot \dot{\vec{e}}_r) \vec{e}_r - \frac{\dot{r}}{r} \vec{e}_r - \dot{\vec{e}}_r + \dot{\vec{e}}_r = 0$$

$$\frac{d\vec{E}}{dt} = \frac{1}{\mu} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right]$$

Useful memo

$$\vec{u} \wedge (\vec{v} \wedge \vec{w}) = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{u} \cdot \vec{v}) \vec{w} \quad \text{and} \quad (\vec{u} \wedge \vec{v}) \wedge \vec{w} = (\vec{u} \cdot \vec{w}) \vec{v} - (\vec{v} \cdot \vec{w}) \vec{u}$$

$$(\vec{u} \wedge \vec{v}) \wedge \vec{w} = \vec{u} \wedge (\vec{v} \wedge \vec{w}) + \vec{v} \wedge (\vec{w} \wedge \vec{u})$$

$$(\vec{u} \wedge \vec{v}) \cdot (\vec{u} \wedge \vec{w}) = u^2 (\vec{v} \cdot \vec{w}) - (\vec{u} \cdot \vec{v})(\vec{u} \cdot \vec{w})$$

$$[= (\vec{u} \wedge \vec{w}) \wedge \vec{u} \cdot \vec{v} = -\vec{u} \wedge (\vec{u} \wedge \vec{w}) \cdot \vec{v} = -\vec{u} \cdot \vec{w} \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \vec{w} \cdot \vec{v} = u^2 (\vec{v} \cdot \vec{w}) - (\vec{u} \cdot \vec{v})(\vec{u} \cdot \vec{w})]$$

The last case transforms cross products into scalar products (so, null in case of orthogonal)

$$\vec{C} = C \vec{e}_w \quad \text{with Euler angles rotation} \quad \overrightarrow{dR}_{Euler} = \frac{d\Omega}{dt} \vec{e}_z + \frac{di}{dt} \vec{e}_{Node} + \frac{dw}{dt} \vec{e}_w$$

$$\frac{d\vec{C}}{dt} = \frac{dC}{dt} \vec{e}_w + \overrightarrow{dR}_{Euler} \wedge \vec{C} = \frac{dC}{dt} \vec{e}_w + C \overrightarrow{dR}_{Euler} \wedge \vec{e}_w = \frac{dC}{dt} \vec{e}_w + C \left(\frac{d\Omega}{dt} \vec{e}_z + \frac{di}{dt} \vec{e}_{Node} + \frac{dw}{dt} \vec{e}_w \right) \wedge \vec{e}_w$$

$$\frac{d\vec{C}}{dt} = \frac{dC}{dt} \vec{e}_w + C \frac{d\Omega}{dt} \vec{e}_z \wedge \vec{e}_w + C \frac{di}{dt} \vec{e}_{Node} \wedge \vec{e}_w$$

$$\frac{d\vec{C}}{dt} \cdot \vec{e}_w = \frac{dC}{dt} \quad \frac{d\vec{C}}{dt} = \vec{r} \wedge \vec{F} \quad \frac{dC}{dt} = \vec{r} \wedge \vec{F} \cdot \vec{e}_w = r \vec{e}_r \wedge \vec{F} \cdot \vec{e}_w \quad \boxed{\frac{dC}{dt} = r \vec{e}_s \cdot \vec{F}} \quad \vec{e}_s = \frac{1}{rC} \vec{C} \wedge \vec{r}$$

$$\frac{d\vec{C}}{dt} \cdot \vec{e}_{Node} = C \frac{d\Omega}{dt} \vec{e}_z \wedge \vec{e}_w \cdot \vec{e}_{Node} = C \frac{d\Omega}{dt} \sin i \quad C \frac{d\Omega}{dt} \sin i = r \vec{e}_r \wedge \vec{F} \cdot \vec{e}_{Node} \quad \boxed{\frac{d\Omega}{dt} = \frac{r \sin(\omega + \nu)}{C \sin i} \vec{e}_w \cdot \vec{F}}$$

$$\text{because} \quad \vec{e}_{Node} \wedge \vec{e}_r = \sin(\omega + \nu) \vec{e}_w \quad \vec{e}_w = \frac{1}{C} \vec{C}$$

$$\frac{d\vec{C}}{dt} \cdot \vec{e}_w \wedge \vec{e}_{Node} = C \frac{di}{dt} \quad \frac{di}{dt} = \frac{r}{C} \vec{e}_r \wedge \vec{F} \cdot \vec{e}_w \wedge \vec{e}_{Node} \quad \frac{di}{dt} = \frac{r}{C} (\vec{e}_w \wedge \vec{e}_{Node}) \wedge \vec{e}_r \cdot \vec{F} \quad \boxed{\frac{di}{dt} = \frac{r \cos(\omega + \nu)}{C} \vec{e}_w \cdot \vec{F}}$$

$\vec{E} = e \vec{e}_{per}$ with \vec{e}_{per} the unit vector focus to perigee

$$\frac{d\vec{E}}{dt} = \frac{de}{dt} \vec{e}_{per} + \overrightarrow{dR}_{Euler} \wedge \vec{E} = \frac{de}{dt} \vec{e}_w + e \overrightarrow{dR}_{Euler} \wedge \vec{e}_{per} = \frac{de}{dt} \vec{e}_w + e \frac{d\Omega}{dt} \vec{e}_z \wedge \vec{e}_{per} + e \frac{di}{dt} \vec{e}_{Node} \wedge \vec{e}_{per} + e \frac{dw}{dt} \vec{e}_w \wedge \vec{e}_{per}$$

$$\frac{d\vec{E}}{dt} \cdot \vec{e}_{per} = \frac{de}{dt} = \frac{1}{\mu} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] \cdot \vec{e}_{per} \quad \boxed{\frac{de}{dt} = \frac{1}{\mu e} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] \cdot \vec{E}}$$

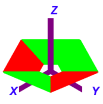
$$\frac{d\vec{E}}{dt} \cdot \vec{e}_w \wedge \vec{e}_{per} = e \frac{d\Omega}{dt} \vec{e}_z \wedge \vec{e}_{per} \cdot \vec{e}_w \wedge \vec{e}_{per} + e \frac{dw}{dt} = e \frac{d\Omega}{dt} \cos i + e \frac{dw}{dt} = \frac{1}{\mu} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] \cdot \vec{e}_w \wedge \vec{e}_{per}$$

$$\vec{e}_z \wedge \vec{e}_{per} \cdot \vec{e}_w \wedge \vec{e}_{per} = (\vec{e}_w \wedge \vec{e}_{per}) \wedge \vec{e}_z \cdot \vec{e}_{per} = -\vec{e}_z \wedge (\vec{e}_w \wedge \vec{e}_{per}) \cdot \vec{e}_{per} = -\vec{e}_z \cdot \vec{e}_{per} \vec{e}_w \cdot \vec{e}_{per} + \vec{e}_z \cdot \vec{e}_w \vec{e}_{per} \cdot \vec{e}_{per} = \vec{e}_z \cdot \vec{e}_w = \cos i$$

$$\boxed{\cos i \frac{d\Omega}{dt} + \frac{dw}{dt} = \frac{1}{\mu C e^2} \left[\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \right] \cdot \vec{C} \wedge \vec{E}}$$

Ref.

[R1] Christophe Koppel, Thrust Orientation Strategies for a Continuous Transfer to GEO, LOTUS conference, ATELIER CNES "TRAJECTOIRES A POUSSEE FAIBLE" 7-8 mars 2000, Toulouse
[R2] Luc Duriez, Cours de Mécanique céleste classique, Laboratoire d'Astronomie de l'Université de Lille 1 et IMCCE de l'Observatoire de Paris, 2002
[R3] Pierre Exertier, Florent Deleflie, Les Equations du Mouvement Orbital Perturbé, Cours de l'Ecole GRGS 2002, Observatoire de la Côte d'Azur (CERGA/URA6527), Av. N. Copernic, F-06130 Grasse, 21 février 2003
[R4] Contact: C. R. Koppel, kci@kopooS.com



Other approaches for Gauss equations development: (ref. 3)

$$\frac{de}{dt} = \frac{1}{\mu e} [\vec{F} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F})] \cdot \left[\frac{1}{\mu} \vec{V} \wedge \vec{C} - \frac{\vec{r}}{r} \right] = \frac{1}{\mu^2 e} (\vec{F} \wedge \vec{C} \cdot \vec{V} \wedge \vec{C} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \cdot \vec{V} \wedge \vec{C}) - \frac{1}{\mu e r} (\vec{F} \wedge \vec{C} \cdot \vec{r} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \cdot \vec{r})$$

$$\vec{F} \wedge \vec{C} \cdot \vec{r} + \vec{V} \wedge (\vec{r} \wedge \vec{F}) \cdot \vec{r} = \vec{r} \wedge \vec{F} \cdot \vec{C} + \vec{r} \wedge \vec{V} \cdot \vec{r} \wedge \vec{F} = 2\vec{C} \cdot \vec{r} \wedge \vec{F}$$

$$\frac{de}{dt} = \frac{C^2}{\mu^2 e} \vec{F} \cdot \vec{V} + \frac{V^2}{\mu^2 e} \vec{C} \cdot \vec{r} \wedge \vec{F} - \frac{2}{\mu e r} \vec{C} \cdot \vec{r} \wedge \vec{F} = \frac{C^2}{\mu^2 e} \vec{F} \cdot \vec{V} + \frac{2}{\mu^2 e} \left(\frac{V^2}{2} - \frac{\mu}{r} \right) \vec{C} \cdot \vec{r} \wedge \vec{F} = \frac{C^2}{\mu^2 e} \vec{F} \cdot \vec{V} + \frac{2}{\mu^2 e} \left(\frac{-\mu}{2a} \right) \vec{C} \cdot \vec{r} \wedge \vec{F}$$

$$\frac{de}{dt} = \frac{C^2}{\mu^2 e} \vec{V} \cdot \vec{F} - \frac{\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu e a}$$

$$e^2 = 1 - \frac{C^2}{\mu a} \quad e^2 = \vec{E} \cdot \vec{E} = 1 - \frac{\vec{C} \cdot \vec{C}}{\mu a} \quad \frac{de}{dt} = \frac{d(\vec{E} \cdot \vec{E})^{1/2}}{dt} = \frac{1}{2} (\vec{E} \cdot \vec{E})^{-1/2} 2\vec{E} \cdot \frac{d\vec{E}}{dt} = \frac{1}{e} \vec{E} \cdot \frac{d\vec{E}}{dt}$$

$$2e \frac{de}{dt} = \frac{d(\vec{C} \cdot \vec{C})}{dt} = \frac{-2\vec{C} \cdot \frac{d\vec{C}}{dt}}{\mu a} + \frac{\vec{C} \cdot \vec{C}}{\mu a^2} \frac{da}{dt} = \frac{-2\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu a} + \frac{C^2}{\mu a^2} \frac{2a^2}{\mu} \vec{V} \cdot \vec{F} = \frac{-2\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu a} + \frac{2C^2}{\mu^2} \vec{V} \cdot \vec{F}$$

$$\frac{de}{dt} = \frac{C^2}{\mu^2 e} \vec{V} \cdot \vec{F} - \frac{\vec{C} \cdot (\vec{r} \wedge \vec{F})}{\mu e a}$$

$\vec{C} \cdot \vec{e}_z = C \cos i$ in the inertial frame, \vec{e}_z is the unit vector along z (fixed vector $\dot{\vec{e}}_z = 0$)

$$\vec{e}_z = \sin(\omega + \nu) \sin i \vec{e}_r + \cos(\omega + \nu) \sin i \vec{e}_s + \cos i \vec{e}_w$$

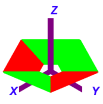
$$\frac{d\vec{C} \cdot \vec{e}_z}{dt} = \frac{d\vec{C}}{dt} \cdot \vec{e}_z = \frac{d(\vec{C} \cdot \vec{C})^{1/2}}{dt} \cos i - C \sin i \frac{di}{dt} = \frac{1}{2} (\vec{C} \cdot \vec{C})^{-1/2} 2\vec{C} \cdot \frac{d\vec{C}}{dt} \cos i - C \sin i \frac{di}{dt}$$

$$\frac{di}{dt} = \frac{\vec{C} \cdot \frac{d\vec{C}}{dt} \cos i - \frac{d\vec{C}}{dt} \cdot \vec{e}_z}{C \sin i} = \frac{\cos i \vec{C} - \vec{e}_z}{C \sin i} \cdot \frac{d\vec{C}}{dt} = \frac{\cos i \vec{e}_w - \vec{e}_z}{C \sin i} \cdot \vec{r} \wedge \vec{F}$$

$$\frac{di}{dt} = \frac{q_{\text{any}} \vec{e}_r + \cos(\omega + \nu) \sin i \vec{e}_s}{-C \sin i} \cdot \vec{r} \wedge \vec{F}, \quad \text{i.e. because the scalar triple product } \vec{e}_r \cdot \vec{r} \wedge \vec{F} = 0$$

$$\frac{di}{dt} = \frac{-\cos(\omega + \nu)}{C} \vec{e}_s \cdot \vec{r} \wedge \vec{F} = \frac{r \cos(\omega + \nu)}{C} \vec{e}_w \cdot \vec{F}$$

$$\frac{di}{dt} = \frac{r \cos(\omega + \nu)}{C} \vec{e}_w \cdot \vec{F}$$



$$\vec{C} = C \sin \Omega \sin i \vec{e}_x - C \cos \Omega \sin i \vec{e}_y + C \cos i \vec{e}_z \quad \text{tg} \Omega = \frac{\vec{C} \cdot \vec{e}_x}{-\vec{C} \cdot \vec{e}_y} \Rightarrow \frac{d \text{tg} \Omega}{dt} = \frac{1}{\cos^2 \Omega} \frac{d\Omega}{dt} \quad \text{and}$$

$$d \frac{\vec{C} \cdot \vec{e}_x}{-\vec{C} \cdot \vec{e}_y} / dt = - \frac{1}{C^2 \cos^2 \Omega \sin^2 i} \frac{d\vec{C}}{dt} \cdot (\vec{C} \cdot \vec{e}_y \vec{e}_x - \vec{C} \cdot \vec{e}_x \vec{e}_y) = \frac{1}{C \cos^2 \Omega \sin i} \frac{d\vec{C}}{dt} \cdot (\cos \Omega \vec{e}_x + \sin \Omega \vec{e}_y)$$

$$\frac{d\Omega}{dt} = \cos^2 \Omega \frac{d \text{tg} \Omega}{dt} = \frac{1}{C \sin i} \frac{d\vec{C}}{dt} \cdot (\cos \Omega \vec{e}_x + \sin \Omega \vec{e}_y) = \frac{1}{C \sin i} \vec{r} \wedge \vec{F} \cdot (\cos \Omega \vec{e}_x + \sin \Omega \vec{e}_y)$$

with $\vec{e}_x = q_x \vec{e}_r + (-\cos(\omega + \nu) \sin \Omega \cos i - \sin(\omega + \nu) \cos \Omega) \vec{e}_s + \sin \Omega \sin i \vec{e}_w$

and $\vec{e}_y = q_y \vec{e}_r + (\cos(\omega + \nu) \cos \Omega \cos i - \sin(\omega + \nu) \sin \Omega) \vec{e}_s - \cos \Omega \sin i \vec{e}_w$

$$q_x = -\sin(\omega + \nu) \sin \Omega \cos i + \cos(\omega + \nu) \cos \Omega$$

$$q_y = \sin(\omega + \nu) \cos \Omega \cos i + \cos(\omega + \nu) \sin \Omega$$

$\cos \Omega \vec{e}_x + \sin \Omega \vec{e}_y = q_{any} \vec{e}_r - \sin(\omega + \nu) \vec{e}_s$; the \vec{e}_r component is cancelled in the scalar triple product

$$\frac{d\Omega}{dt} = \frac{-\sin(\omega + \nu)}{C \sin i} \vec{e}_s \cdot \vec{r} \wedge \vec{F} = \frac{r \sin(\omega + \nu)}{C \sin i} \vec{e}_w \cdot \vec{F}$$

$$\frac{d\Omega}{dt} = \frac{r \sin(\omega + \nu)}{C \sin i} \vec{e}_w \cdot \vec{F}$$

Case of circular or/and equatorial orbits

...
