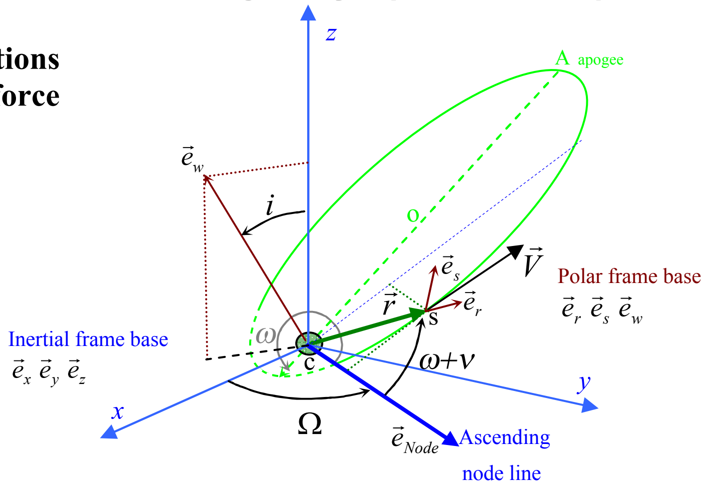
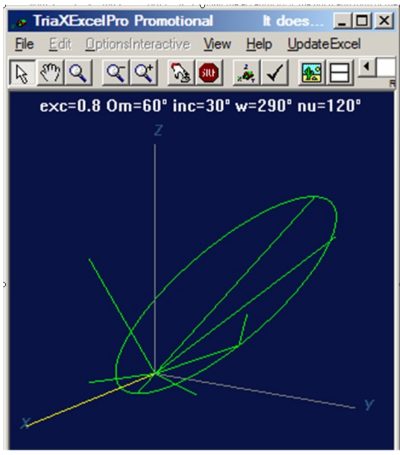
Trajectory equations under a central force





The trajectory under a central force $\ddot{\vec r} = \frac{-\mu}{r^3}\vec r$ is fixed at each time *t* by the vectors $\vec r(t), \vec V(t)$.

With $\vec r(t), \vec V(t)$ one can derive the 5 fixed elliptical parameters: a, e, i, Ω, ω and one time dependant $\nu(t)$

It is fully known that the vis viva equation (orbital energy conservation) is $\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a}$ which give the semi-major axis *a*.

With $\vec C = \vec r \wedge \vec V$ and $C^2 = \mu a(1-e^2)$ giving the eccentricity *e*, the polar equation of the ellipse $r(1+e\cos\nu) = a(1-e^2)$ and its radial derivative $\mu e r \sin\nu = C\,\vec V\cdot\vec r$ give the true anomaly $\nu(t)$. The angles of inclination $\in[0, \pi[$ and right ascension of the ascending node i, Ω are defined by the vector $\vec C$ ($\vec C\cdot\vec e_z = C\cos i$; $C\sin\Omega = \vec C\cdot\vec e_x$; $C\cos\Omega = -\vec C\cdot\vec e_y$). The perigee argument ω is given by the polar equation projected on the node line and then on the inertial frame:

$$\vec r\cdot\vec e_x = -r\sin(\omega+\nu)\cos i\sin\Omega + r\cos(\omega+\nu)\cos\Omega \qquad \vec r\cdot\vec e_y = +r\sin(\omega+\nu)\cos i\cos\Omega + r\cos(\omega+\nu)\sin\Omega$$

$$\vec r\cdot\vec e_z = +r\sin(\omega+\nu)\sin i \;\Rightarrow\; \vec r\cdot\vec e_x\cos\Omega + \vec r\cdot\vec e_y\sin\Omega = r\cos(\omega+\nu) \text{ and } \quad r\sin(\omega+\nu) = \vec r\cdot\vec e_z/\sin i$$

The mean anomaly is $M(t) = \sqrt{\mu/a^3}\,(t-t_{per}) = \sqrt{\mu/a^3}\,(u(t) - e\sin u(t))$ where the time when crossing the perigee is t_{per} and *u* is the eccentric anomaly (angle from center *O* wrt perigee) $r = a(1-e\cos u(t))$.

Trajectory equations with perturbation force $\vec F$ force per unit of mass or acceleration $\ddot{\vec r} = \frac{d\vec V}{dt} = \frac{-\mu}{r^3}\vec r + \vec F$

$$\vec C = \vec r\wedge\vec V \qquad \frac{d\vec C}{dt} = \vec r\wedge\vec F \qquad h = \frac{V^2}{2} - \frac{\mu}{r} \qquad \frac{dh}{dt} = \vec V\cdot\vec F \qquad \vec E = \frac{1}{\mu}\vec V\wedge\vec C - \frac{\vec r}{r} \qquad \frac{d\vec E}{dt} = \frac{1}{\mu}\left[\vec F\wedge\vec C + \vec V\wedge(\vec r\wedge\vec F)\right]$$

$$\frac{V^2}{2} - \frac{\mu}{r} = \frac{-\mu}{2a} \qquad \frac{da}{dt} = \frac{2a^2}{\mu}\vec V\cdot\vec F \qquad \frac{dC}{dt} = \frac{1}{C}\vec C\wedge\vec r\cdot\vec F$$

$$e^2 = 1 - \frac{C^2}{\mu a} \qquad \frac{de}{dt} = \frac{C^2}{\mu^2 e}\vec V\cdot\vec F - \frac{\vec C\cdot\left(\vec r\wedge\vec F\right)}{\mu e a} \qquad \frac{de}{dt} = \frac{1}{\mu e}\left[\vec F\wedge\vec C + \vec V\wedge(\vec r\wedge\vec F)\right]\cdot\vec E$$

$$\vec C\cdot\vec e_z = C\cos i \qquad \frac{di}{dt} = \frac{r\cos(\omega+\nu)}{C^2}\vec C\cdot\vec F$$

$$\mathrm{tg}\Omega = \frac{\vec C\cdot\vec e_x}{-\vec C\cdot\vec e_y} \qquad \frac{d\Omega}{dt} = \frac{r\sin(\omega+\nu)}{C^2\sin i}\vec C\cdot\vec F$$

$$\frac{d\omega}{dt} = \frac{1}{\mu C e^2}\left[\vec F\wedge\vec C + \vec V\wedge(\vec r\wedge\vec F)\right]\cdot\vec C\wedge\vec E - \cos i\frac{d\Omega}{dt}$$

$$\frac{dM}{dt} \approx M(t)\Big/\left(t - t_{per}\right) + \frac{-1}{\sqrt{\mu a}}\left[2\vec r\cdot\vec F + C\frac{d\omega}{dt} + C\cos i\frac{d\Omega}{dt}\right]$$

Application: best command to increase the perigee, [R1]

$$r_{per} = a(1-e) \qquad \frac{dr_{per}}{dt} = (1-e)\dot{a} - a\dot{e}$$

$$\frac{dr_{per}}{dt} = \frac{\vec{C}\cdot(\vec{r}\wedge\vec{F})}{\mu e} - \frac{(1-e)^2 a^2}{\mu e}\vec{V}\cdot\vec{F}$$

To increase the perigee at best, the thrust shall be in the local horizontal (i.e. along $\vec{C}\wedge\vec{r}$ in order to get $\vec{C}\cdot(\vec{r}\wedge\vec{F})$ maximum) and around apogee (i.e. $\vec{r}$ maximum and $\vec{V}\cdot\vec{F}$ minimum).

Application: best command to increase the apogee, [R1]

$$r_{apo} = a(1+e) \qquad \frac{dr_{apo}}{dt} = (1+e)\dot{a} + a\dot{e}$$

$$\frac{dr_{apo}}{dt} = \frac{(1+e)^2 a^2}{\mu e}\vec{V}\cdot\vec{F} - \frac{\vec{C}\cdot(\vec{r}\wedge\vec{F})}{\mu e}$$

To increase the apogee at best, the thrust shall be along velocity (i.e. $\vec{V}\cdot\vec{F}$ maximum) and around perigee (i.e. $\vec{V}$ maximum and $\vec{C}\cdot(\vec{r}\wedge\vec{F})$ minimum).

Development of Gauss equations [R2] , [R3] , [R4]

Note: all derivative wrt an inertial frame, thus $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are fixed vector $\Rightarrow \dot{\vec{e}}_x, \dot{\vec{e}}_y, \dot{\vec{e}}_z = 0$

$$\ddot{\vec{r}} = \frac{d\vec{V}}{dt} = \frac{-\mu}{r^3}\vec{r} + \vec{F} \qquad \text{nota : } \vec{V} = \frac{d\overrightarrow{cs}}{dt} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \frac{d(r\vec{e}_r)}{dt} = \frac{dr}{dt}\vec{e}_r + \mathrm{r}\frac{d\vec{e}_r}{dt} = \dot{r}\vec{e}_r + \mathrm{r}\dot{\vec{e}}_r = \dot{r}\vec{e}_r + \mathrm{r}\dot{v}\,\vec{e}_s$$

$$\vec{C} = \vec{r}\wedge\vec{V} \qquad \frac{d\vec{C}}{dt} = \frac{d\vec{r}}{dt}\wedge\vec{V} + \vec{r}\wedge\frac{d\vec{V}}{dt} = \vec{V}\wedge\vec{V} + \vec{r}\wedge\frac{-\mu}{r^3}\vec{r} + \vec{r}\wedge\vec{F} = 0 + 0 + \vec{r}\wedge\vec{F} \qquad \boxed{\frac{d\vec{C}}{dt} = \vec{r}\wedge\vec{F}}$$

$$h = \frac{V^2}{2} - \frac{\mu}{r} \qquad \text{for elliptical orbits} \quad h = \frac{-\mu}{2a}$$

$$\text{with} \qquad \frac{1}{2}\frac{d\dot{r}^2}{dt} = \ddot{\vec{r}}\cdot\dot{\vec{r}} \qquad and \qquad \frac{d\frac{\mu}{r}}{dt} = \frac{d\mu(\vec{r}\cdot\vec{r})^{-1/2}}{dt} = \mu\frac{-1}{2}(\vec{r}\cdot\vec{r})^{-3/2}.2(\vec{r}\cdot\dot{\vec{r}}) = \frac{-\mu}{r^3}\vec{r}\cdot\dot{\vec{r}}$$

$$\ddot{\vec{r}}\cdot\dot{\vec{r}} = \left(\frac{-\mu}{r^3}\vec{r} + \vec{F}\right)\cdot\dot{\vec{r}} = \frac{-\mu}{r^3}\vec{r}\cdot\dot{\vec{r}} + \vec{F}\cdot\dot{\vec{r}} = \frac{d\frac{\mu}{r}}{dt} + \vec{F}\cdot\dot{\vec{r}} = \frac{d\frac{\mu}{r}}{dt} + \vec{V}\cdot\vec{F}$$

$$\frac{dh}{dt} = \frac{d\left(\frac{\dot{r}^2}{2} - \frac{\mu}{r}\right)}{dt} = \frac{1}{2}\frac{d\dot{r}^2}{dt} - \frac{d\frac{\mu}{r}}{dt} = \vec{V}\cdot\vec{F}$$

$$\boxed{\frac{dh}{dt} = \vec{V}\cdot\vec{F}}$$

$$a = \frac{-\mu}{2h}$$

$$\frac{da}{dt} = \frac{\mu}{2h^2}.\frac{dh}{dt} = \frac{\mu}{2h^2}\vec{V}\cdot\vec{F} = \frac{2a^2}{\mu}\vec{V}\cdot\vec{F} \qquad \boxed{\frac{da}{dt} = \frac{2a^2}{\mu}\vec{V}\cdot\vec{F}}$$

$$\vec{E}=\frac{1}{\mu}\vec{V}\wedge\vec{C}-\frac{\vec{r}}{r}\qquad \frac{d\vec{E}}{dt}=\frac{1}{\mu}\frac{d\vec{V}}{dt}\wedge\vec{C}+\frac{1}{\mu}\vec{V}\wedge\frac{d\vec{C}}{dt}-\dot{\vec{e}}_r=\frac{1}{\mu}\left[\frac{-\mu}{r^3}\vec{r}+\vec{F}\right]\wedge\vec{C}+\frac{1}{\mu}\vec{V}\wedge(\vec{r}\wedge\vec{F})-\dot{\vec{e}}_r$$

$$\frac{d\vec{E}}{dt}=\frac{1}{\mu}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]-\left(\frac{\vec{r}}{r^3}\wedge(\vec{r}\wedge\vec{V})+\dot{\vec{e}}_r\right)=\frac{1}{\mu}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]+0\qquad\text{because}$$

$$\frac{\vec{r}}{r^3}\wedge(\vec{r}\wedge\vec{V})+\dot{\vec{e}}_r=\frac{\vec{r}\cdot\vec{V}}{r^3}\vec{r}-\frac{\vec{r}\cdot\vec{r}}{r^3}\vec{V}+\dot{\vec{e}}_r=\frac{\vec{e}_r\cdot\vec{V}}{r}\vec{e}_r-\frac{\vec{V}}{r}+\dot{\vec{e}}_r=\frac{\dot{r}}{r}\vec{e}_r+\underbrace{\cancel{(\vec{e}_r\cdot\dot{\vec{e}}_r)}}_{=0}\vec{e}_r-\frac{\dot{r}}{r}\vec{e}_r-\dot{\vec{e}}_r+\dot{\vec{e}}_r=0$$

$$\frac{d\vec{E}}{dt}=\frac{1}{\mu}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]$$

Useful memo

$\vec{u}\wedge(\vec{v}\wedge\vec{w})=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}$ and $(\vec{u}\wedge\vec{v})\wedge\vec{w}=(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}$

$(\vec{u}\wedge\vec{v})\wedge\vec{w}=\vec{u}\wedge(\vec{v}\wedge\vec{w})+\vec{v}\wedge(\vec{w}\wedge\vec{u})$

$(\vec{u}\wedge\vec{v})\cdot(\vec{u}\wedge\vec{w})=u^2(\vec{v}\cdot\vec{w})-(\vec{u}\cdot\vec{v})(\vec{u}\cdot\vec{w})$

$\left[=(\vec{u}\wedge\vec{w})\wedge\vec{u}\cdot\vec{v}=-\vec{u}\wedge(\vec{u}\wedge\vec{w})\cdot\vec{v}=-\vec{u}.\vec{w}\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{u}\vec{w}\cdot\vec{v}=u^2(\vec{v}\cdot\vec{w})-(\vec{u}\cdot\vec{v})(\vec{u}\cdot\vec{w})\right]$

The last case transforms cross products into scalar products (so, null in case of orthogonal)

$$\vec{C}=C\vec{e}_w\quad\text{with Euler angles rotation }\overrightarrow{dR_{Euler}}=\frac{d\Omega}{dt}\vec{e}_z+\frac{di}{dt}\vec{e}_{Node}+\frac{dw}{dt}\vec{e}_w$$

$$\frac{d\vec{C}}{dt}=\frac{dC}{dt}\vec{e}_w+\overrightarrow{dR_{Euler}}\wedge\vec{C}=\frac{dC}{dt}\vec{e}_w+C\,\overrightarrow{dR_{Euler}}\wedge\vec{e}_w=\frac{dC}{dt}\vec{e}_w+C\left(\frac{d\Omega}{dt}\vec{e}_z+\frac{di}{dt}\vec{e}_{Node}+\frac{dw}{dt}\vec{e}_w\right)\wedge\vec{e}_w$$

$$\frac{d\vec{C}}{dt}=\frac{dC}{dt}\vec{e}_w+C\frac{d\Omega}{dt}\vec{e}_z\wedge\vec{e}_w+C\frac{di}{dt}\vec{e}_{Node}\wedge\vec{e}_w$$

$$\frac{d\vec{C}}{dt}\cdot\vec{e}_w=\frac{dC}{dt}\qquad\frac{d\vec{C}}{dt}=\vec{r}\wedge\vec{F}\qquad\frac{dC}{dt}=\vec{r}\wedge\vec{F}\cdot\vec{e}_w=r\vec{e}_r\wedge\vec{F}\cdot\vec{e}_w\qquad\boxed{\frac{dC}{dt}=r\vec{e}_s\cdot\vec{F}}\qquad\vec{e}_s=\frac{1}{rC}\vec{C}\wedge\vec{r}$$

$$\frac{d\vec{C}}{dt}\cdot\vec{e}_{Node}=C\frac{d\Omega}{dt}\vec{e}_z\wedge\vec{e}_w\cdot\vec{e}_{Node}=C\frac{d\Omega}{dt}\sin i\qquad C\frac{d\Omega}{dt}\sin i=r\vec{e}_r\wedge\vec{F}\cdot\vec{e}_{Node}\qquad\boxed{\frac{d\Omega}{dt}=\frac{r\sin(\omega+\nu)}{C\sin i}\vec{e}_w\cdot\vec{F}}$$

$$\text{because}\quad\vec{e}_{Node}\wedge\vec{e}_r=\sin(\omega+\nu)\vec{e}_w\qquad\qquad\vec{e}_w=\frac{1}{C}\vec{C}$$

$$\frac{d\vec{C}}{dt}\cdot\vec{e}_w\wedge\vec{e}_{Node}=C\frac{di}{dt}\qquad\frac{di}{dt}=\frac{r}{C}\vec{e}_r\wedge\vec{F}\cdot\vec{e}_w\wedge\vec{e}_{Node}\qquad\frac{di}{dt}=\frac{r}{C}(\vec{e}_w\wedge\vec{e}_{Node})\wedge\vec{e}_r\cdot\vec{F}\qquad\boxed{\frac{di}{dt}=\frac{r\cos(\omega+\nu)}{C}\vec{e}_w\cdot\vec{F}}$$

$\vec{E}=e\vec{e}_{per}$ with $\vec{e}_{per}$ the unit vector focus to perigee

$$\frac{d\vec{E}}{dt}=\frac{de}{dt}\vec{e}_{per}+\overrightarrow{dR_{Euler}}\wedge\vec{E}=\frac{de}{dt}\vec{e}_w+e\,\overrightarrow{dR_{Euler}}\wedge\vec{e}_{per}=\frac{de}{dt}\vec{e}_w+e\frac{d\Omega}{dt}\vec{e}_z\wedge\vec{e}_{per}+e\frac{di}{dt}\vec{e}_{Node}\wedge\vec{e}_{per}+e\frac{dw}{dt}\vec{e}_w\wedge\vec{e}_{per}$$

$$\frac{d\vec{E}}{dt}\cdot\vec{e}_{per}=\frac{de}{dt}=\frac{1}{\mu}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]\cdot\vec{e}_{per}\qquad\boxed{\frac{de}{dt}=\frac{1}{\mu e}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]\cdot\vec{E}}$$

$$\frac{d\vec{E}}{dt}\cdot\vec{e}_w\wedge\vec{e}_{per}=e\frac{d\Omega}{dt}\vec{e}_z\wedge\vec{e}_{per}\cdot\vec{e}_w\wedge\vec{e}_{per}+e\frac{dw}{dt}=e\frac{d\Omega}{dt}\cos i+e\frac{dw}{dt}=\frac{1}{\mu}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]\cdot\vec{e}_w\wedge\vec{e}_{per}$$

$$\vec{e}_z\wedge\vec{e}_{per}\cdot\vec{e}_w\wedge\vec{e}_{per}=(\vec{e}_w\wedge\vec{e}_{per})\wedge\vec{e}_z\cdot\vec{e}_{per}=-\vec{e}_z\wedge(\vec{e}_w\wedge\vec{e}_{per})\cdot\vec{e}_{per}=-\vec{e}_z.\vec{e}_{per}\vec{e}_w\cdot\vec{e}_{per}+\vec{e}_z.\vec{e}_w\vec{e}_{per}\cdot\vec{e}_{per}=\vec{e}_z.\vec{e}_w=\cos i$$

$$\boxed{\cos i\frac{d\Omega}{dt}+\frac{dw}{dt}=\frac{1}{\mu Ce^2}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]\cdot\vec{C}\wedge\vec{E}}$$

Ref.

[R1] Christophe Koppel, Thrust Orientation Strategies for a Continuous Transfer to GEO, LOTUS conference, ATELIER CNES "TRAJECTOIRES A POUSSEE FAIBLE" 7-8 mars 2000, Toulouse
[R2] Luc Duriez, Cours de Mécanique céleste classique, Laboratoire d'Astronomie de l'Université de Lille 1 et IMCCE de l'Observatoire de Paris, 2002
[R3] Pierre Exertier, Florent Deleflie, Les Equations du Mouvement Orbital Perturbé, Cours de l'Ecole GRGS 2002, Observatoire de la Côte d'Azur (CERGA/URA6527), Av. N. Copernic, F-06130 Grasse, 21 février 2003
[R4] Contact: C. R. Koppel, kci@kopoos.com

Other approaches for Gauss equations development: (ref. 3)

$$\frac{de}{dt}=\frac{1}{\mu e}\left[\vec{F}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\right]\cdot\left[\frac{1}{\mu}\vec{V}\wedge\vec{C}-\frac{\vec{r}}{r}\right]=\frac{1}{\mu^2 e}\left(\vec{F}\wedge\vec{C}\cdot\vec{V}\wedge\vec{C}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\cdot\vec{V}\wedge\vec{C}\right)-\frac{1}{\mu e r}\left(\vec{F}\wedge\vec{C}\cdot\vec{r}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\cdot\vec{r}\right)$$

$$\vec{F}\wedge\vec{C}\cdot\vec{r}+\vec{V}\wedge(\vec{r}\wedge\vec{F})\cdot\vec{r}=\vec{r}\wedge\vec{F}\cdot\vec{C}+\vec{r}\wedge\vec{V}\cdot\vec{r}\wedge\vec{F}=2\vec{C}\cdot\vec{r}\wedge\vec{F}$$

$$\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec{F}\cdot\vec{V}+\frac{V^2}{\mu^2 e}\vec{C}\cdot\vec{r}\wedge\vec{F}-\frac{2}{\mu e r}\vec{C}\cdot\vec{r}\wedge\vec{F}=\frac{C^2}{\mu^2 e}\vec{F}\cdot\vec{V}+\frac{2}{\mu^2 e}\left(\frac{V^2}{2}-\frac{\mu}{r}\right)\vec{C}\cdot\vec{r}\wedge\vec{F}=\frac{C^2}{\mu^2 e}\vec{F}\cdot\vec{V}+\frac{2}{\mu^2 e}\left(\frac{-\mu}{2a}\right)\vec{C}\cdot\vec{r}\wedge\vec{F}$$

$$\boxed{\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec{V}\cdot\vec{F}-\frac{\vec{C}\cdot\left(\vec{r}\wedge\vec{F}\right)}{\mu e a}}$$

$$e^2=1-\frac{C^2}{\mu a}\qquad e^2=\vec{E}\cdot\vec{E}=1-\frac{\vec{C}\cdot\vec{C}}{\mu a}\qquad \frac{de}{dt}=\frac{d\left(\vec{E}\cdot\vec{E}\right)^{1/2}}{dt}=\frac{1}{2}\left(\vec{E}\cdot\vec{E}\right)^{-1/2}2\vec{E}\cdot\frac{d\vec{E}}{dt}=\frac{1}{e}\vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$2e\frac{de}{dt}=\frac{d\text{-}\dfrac{\vec{C}\cdot\vec{C}}{\mu a}}{dt}=\frac{-2\vec{C}\cdot\dfrac{d\vec{C}}{dt}}{\mu a}+\frac{\vec{C}\cdot\vec{C}}{\mu a^2}\frac{da}{dt}=\frac{-2\vec{C}\cdot\left(\vec{r}\wedge\vec{F}\right)}{\mu a}+\frac{C^2}{\mu a^2}\frac{2a^2}{\mu}\vec{V}\cdot\vec{F}=\frac{-2\vec{C}\cdot\left(\vec{r}\wedge\vec{F}\right)}{\mu a}+\frac{2C^2}{\mu^2}\vec{V}\cdot\vec{F}$$

$$\frac{de}{dt}=\frac{C^2}{\mu^2 e}\vec{V}\cdot\vec{F}-\frac{\vec{C}\cdot\left(\vec{r}\wedge\vec{F}\right)}{\mu e a}$$

$\vec{C}\cdot\vec{e}_z=C\cos i$ in the inertial frame, $\vec{e}_z$ is the unit vector along z (fixed vector $\dot{\vec{e}}_z=0$)

$$\vec{e}_z=\sin(w+v)\sin i\,\vec{e}_r+\cos(w+v)\sin i\,\vec{e}_s+\cos i\,\vec{e}_w$$

$$\frac{d\vec{C}\cdot\vec{e}_z}{dt}=\frac{d\vec{C}}{dt}\cdot\vec{e}_z=\frac{d\left(\vec{C}\cdot\vec{C}\right)^{\frac{1}{2}}}{dt}\cos i-C\sin i\frac{di}{dt}=\frac{1}{2}\left(\vec{C}\cdot\vec{C}\right)^{\frac{-1}{2}}2\vec{C}\cdot\frac{d\vec{C}}{dt}\cos i-C\sin i\frac{di}{dt}$$

$$\frac{di}{dt}=\frac{\dfrac{\vec{C}}{C}\cdot\dfrac{d\vec{C}}{dt}\cos i-\dfrac{d\vec{C}}{dt}\cdot\vec{e}_z}{C\sin i}=\frac{\dfrac{\cos i}{C}\vec{C}-\vec{e}_z}{C\sin i}\cdot\frac{d\vec{C}}{dt}=\frac{\cos i\vec{e}_w-\vec{e}_z}{C\sin i}\cdot\vec{r}\wedge\vec{F}$$

$$\frac{di}{dt}=\frac{q_{\text{any}}\vec{e}_r+\cos(\omega+v)\sin i\,\vec{e}_s}{-C\sin i}\cdot\vec{r}\wedge\vec{F},\quad \text{i.e. because the scalar triple product } \vec{e}_r\cdot\vec{r}\wedge\vec{F}=0$$

$$\frac{di}{dt}=\frac{-\cos(\omega+v)}{C}\vec{e}_s\cdot\vec{r}\wedge\vec{F}=\frac{r\cos(\omega+v)}{C}\vec{e}_w\cdot\vec{F}$$

$$\frac{di}{dt}=\frac{r\cos(\omega+v)}{C}\vec{e}_w\cdot\vec{F}$$

$$\vec{C} = C\sin\Omega\sin i\vec{e}_x - C\cos\Omega\sin i\vec{e}_y + C\cos i\vec{e}_z \qquad \mathrm{tg}\Omega = \frac{\vec{C}\cdot\vec{e}_x}{-\vec{C}\cdot\vec{e}_y} \Rightarrow \frac{d\mathrm{tg}\Omega}{dt} = \frac{1}{\cos^2\Omega}\frac{d\Omega}{dt} \quad \text{and}$$

$$\mathrm{d}\frac{\vec{C}\cdot\vec{e}_x}{-\vec{C}\cdot\vec{e}_y}\Big/ dt = -\frac{1}{C^2\cos^2\Omega\sin^2 i}\frac{d\vec{C}}{dt}\cdot(\vec{C}\cdot\vec{e}_y\vec{e}_x - \vec{C}\cdot\vec{e}_x\vec{e}_y) = \frac{1}{C\cos^2\Omega\sin i}\frac{d\vec{C}}{dt}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y)$$

$$\frac{d\Omega}{dt} = \cos^2\Omega\frac{d\mathrm{tg}\Omega}{dt} = \frac{1}{C\sin i}\frac{d\vec{C}}{dt}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y) = \frac{1}{C\sin i}\vec{r}\wedge\vec{F}\cdot(\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y)$$

with $\vec{e}_x = q_x\vec{e}_r + \left(-\cos(\omega+v)\sin\Omega\cos i - \sin(\omega+v)\cos\Omega\right)\vec{e}_s + \sin\Omega\sin i\vec{e}_w$

and $\vec{e}_y = q_y\vec{e}_r + \left(+\cos(\omega+v)\cos\Omega\cos i - \sin(\omega+v)\sin\Omega\right)\vec{e}_s - \cos\Omega\sin i\vec{e}_w$

$$q_x = -\sin(\omega+v)\sin\Omega\cos i + \cos(\omega+v)\cos\Omega$$

$$q_y = \sin(\omega+v)\cos\Omega\cos i + \cos(\omega+v)\sin\Omega$$

$\cos\Omega\vec{e}_x + \sin\Omega\vec{e}_y = q_{any}\vec{e}_r - \sin(\omega+v)\vec{e}_s$; the $\vec{e}_r$ component is cancelled in the scalar triple product

$$\frac{d\Omega}{dt} = \frac{-\sin(\omega+v)}{C\sin i}\vec{e}_s\cdot\vec{r}\wedge\vec{F} = \frac{r\sin(\omega+v)}{C\sin i}\vec{e}_w\cdot\vec{F}$$

$$\frac{d\Omega}{dt} = \frac{r\sin(\omega+v)}{C\sin i}\vec{e}_w\cdot\vec{F}$$

Case of circular or/and equatorial orbits

…
